

# A Set Theoretic Approach for Knowledge Representation: the Representation Part

Yi Zhou

Artificial Intelligence Research Group  
Western Sydney University, NSW Australia

March 14, 2016

## Abstract

In this paper, we propose a set theoretic approach for knowledge representation. While the syntax of an application domain is captured by set theoretic constructs including individuals, concepts and operators, knowledge is formalized by equality assertions. We first present a primitive form that uses minimal assumed knowledge and constructs. Then, assuming naive set theory, we extend it by definitions, which are special kinds of knowledge. Interestingly, we show that the primitive form is expressive enough to define logic operators, not only propositional connectives but also quantifiers.

## 1 Introduction

Knowledge representation and reasoning (KR) is one of the central focuses of Artificial Intelligence (AI) [Baral, 2003; Brachman and Levesque, 2004; van Harmelen *et al.*, 2008]. KR intends to syntactically formalize information in an application domain as knowledge. Then, complex problems in the domain can be solved by reasoning about the knowledge. KR is not only of its own interests but also highly influential to many other subfields in AI, including expert systems, multi-agent systems, planning, uncertainty, cognitive robotics and semantic Web [Brachman and Levesque, 2004; van Harmelen *et al.*, 2008].

Nevertheless, KR has encountered huge difficulties. One issue is that there are too many features and building blocks to be considered [van Harmelen *et al.*, 2008], for instance, propositions, variables, connectives, rules, actions, common sense, time/space, uncertainty, mental states and so on. In fact, KR has made huge successes on formalizing these building blocks separately. However, combing them together, even several of them, seems an extremely difficult task. On the other side, many application domains, e.g. robotics, need multiple building blocks at the same time.

Another critical related issue is about the balance between expressiveness and efficiency. It is widely believe that the more expressive, the less efficient, and vice versa [Levesque and Brachman, 1987]. However, in many application domains, e.g., robotics, we need both. Yet this is a very difficult task for KR formalisms. Consider

propositional logic, a fundamental KR formalism that only takes propositions and natural propositional connectives into account. The inference problem is coNP complete, which means that, most likely, it takes exponential time in worst case.

Against this backdrop, we argue that KR needs a simple, extensible, expressive and efficient approach. “Extensible” means that this approach should allow a current system in it to be easily extended with more building blocks. “Expressive” means that this approach should be able to represent different types of knowledge for a given set of building blocks. “Efficient” means that this approach can efficiently reason about and derive new knowledge in order to solve problems. Nevertheless, “simple” is an ambiguous term, which can be further elaborated into three aspects: primitive — using minimal primitive constructs, succinct — able to represent knowledge in various application domains with relatively small knowledge bases and user-friendly — easy to be understood and used by knowledge engineers.

Towards this goal, we propose a set theoretic knowledge representation approach for syntactically representing knowledge in application domains. While the syntax of an application domain is captured by a set of individuals, concepts and operators, knowledge is simply formalized by equality assertions of the form  $a = b$ , where  $a$  and  $b$  are either atomic individuals or compound individuals. Semantically, individuals, concepts and operators are interpreted as elements, sets and functions respectively in naive set theory and knowledge of the form  $a = b$  means that the two individuals  $a$  and  $b$  are referring to the same element.

We separate representation and reasoning. This paper is mainly concerned with the basic ideas and the representation part, and we leave the reasoning part to another paper. In this paper, we first present the primitive form that uses minimal assumed knowledge and primitive constructs. Then, assuming naive set theory, we extend it with more building blocks by definitions that use assertions to define new syntactic objects, including individuals, concepts and operators. Once these new objects are defined, they can be used as a basis to define more. As an example, we show that we can define multi-assertions by using Cartesian product, and nested assertions by using multi-assertions. Interestingly, we show that this method, i.e., extending the primitive form by definitions based on naive set theory, is powerful enough to syntactically capture logic operators, including both propositional connectives and quantifiers.

## 2 The Primitive Form

In this section, we present the primitive form of our set theoretic knowledge representation approach. As the goal is to syntactically represent knowledge in application domains, there are two essential tasks, i.e., how to capture the syntax of the domain and how to represent knowledge in it.

We assume naive set theory [Halmos, 1960], including the basic concepts such as elements, sets and functions, Cartesian product, the built-in relationships  $\in$  and  $\subseteq$ , the built-in operators  $\cup$ ,  $\cap$  and  $\setminus$ , the Boolean set  $\{\top, \perp\}$  and the set  $\mathbb{N}$  of natural numbers, cardinality and set specifications.

## 2.1 Capturing the syntax

Given an application domain, a *syntactic structure* (*structure* for short if clear from the context) of the domain is a triple  $\langle \mathcal{I}, \mathcal{C}, \mathcal{O} \rangle$ , where  $\mathcal{I}$  is a collection of *individuals*, representing objects in the domain,  $\mathcal{C} \subseteq 2^{\mathcal{I}}$  a collection of *concepts*, representing groups of objects sharing something in common and  $\mathcal{O}$  a collection of *operators*, representing relationships and connections among individuals and concepts.

Concepts and operators can be considered as individuals as well. If needed, we can have concepts of concepts, concepts of operators, concepts of concepts of operators and so on.

An operator could be multi-ary, that is, it maps a tuple of individuals into a single individual.<sup>1</sup> Each multi-ary operator  $O$  is associated with a *domain* of the form  $(C_1, \dots, C_n)$ , representing all possible values that the operator  $O$  can operate on, where  $C_i, 1 \leq i \leq n$ , is a concept. We call  $n$  the *arity* of  $O$ . For a tuple  $(a_1, \dots, a_n)$  matching the domain of an operator  $O$ , i.e.,  $a_i \in C_i, 1 \leq i \leq n$ ,  $O$  maps  $(a_1, \dots, a_n)$  into an individual, denoted by  $O(a_1, \dots, a_n)$ . We also use  $O(C_1, \dots, C_n)$  to denote the set  $\{O(a_1, \dots, a_n) \mid a_i \in C_i\}$ , called the *range* of the operator  $O$ .

Operators are similar to functions in first-order logic but differs in two essential ways. First, operators are many-sorted as  $C_1, \dots, C_n$  could be different concepts. More importantly,  $C_1, \dots, C_n$  could be high-order constructs, e.g., assertions, concepts of concepts, concepts of operators and so on.

For instance, consider the arithmetic domain, in which 0, 1, 2, etc., are individuals; the set  $\mathbb{N}$  of natural numbers is a concept; the successor operator *Succ* and the add operators *Add* are operators.

For convenience, if  $O$  is unary, we sometimes use  $a.O$  ( $C.O$ ) to denote  $O(a)$  ( $O(C)$ ), where  $a \in \mathcal{I}$  and  $C \in \mathcal{C}$ . If  $O$  is binary, we sometimes use  $a O b$  ( $A O B$ ) to denote  $O(a, b)$  ( $O(A, B)$ ), where  $a, b \in \mathcal{I}$  and  $A, B \in \mathcal{C}$ . If the range of an operator  $O$  is Boolean, we sometimes use  $O(a_1, \dots, a_n)$  to denote  $O(a_1, \dots, a_n) = \top$ .

## 2.2 Representing knowledge

Let  $\langle \mathcal{I}, \mathcal{C}, \mathcal{O} \rangle$  be a syntactic structure. A *term* is an individual, either an atomic individual  $a \in \mathcal{I}$  or the result  $O(a_1, \dots, a_n)$  of an operator  $O$  operating on some individuals  $a_1, \dots, a_n$ . We also call the latter *compound individuals*.

An *assertion* is of the form

$$a = b, \tag{1}$$

where  $a$  and  $b$  are two terms. Intuitively, an assertion of the form (1) is a piece of knowledge in the application domain, claiming that the left and right side are referring to the same objects. Here,  $=$  is the built-in equality relation in naive set theory. Hence,  $a = b$  can be understood in alternative way that  $=(a, b)$  is true. A *knowledge base* is a set of assertions. Terms and assertions can be considered as individuals as well.

For instance, in arithmetic,  $0 = \text{Succ}(1)$  and  $2 + 3 = 5$  are two typical assertions.

---

<sup>1</sup>Note that in naive set theory, a tuple of sets is a Cartesian product of some sets, which itself is a set as well. Therefore, multi-ary operators can essentially be viewed as single-ary.

Similar to concepts that group individuals, we use schemas to group terms and assertions. A *schema term* is either an atomic concept  $C \in \mathcal{C}$  or the collection of results  $O(C_1, \dots, C_n)$ . Essentially, a schema term represents a set of terms, in which every concept is grounded by a corresponding individual. Then, a *schema assertion* is of the same form as form (1) except that terms can be replaced by schema terms. Similarly, a schema assertion represents a set of assertions.

Note that it could be the case that two or more different individuals are referring to the same concept  $C$  in schema terms and assertions. In this case, we need to use different *copies* of  $C$ , denoted by  $C^1, C^2, \dots$ , to distinguish among them. For instance, all assertions  $x = y$ , where  $x$  and  $y$  are numbers, are captured by the schema assertion  $\mathbb{N}^1 = \mathbb{N}^2$ . On the other side, in a schema, the same copy of a concept  $C$  can only refer to the same individual. For instance,  $\mathbb{N} = \mathbb{N}$  is the set of all assertions of the form  $x = x$ , where  $x \in \mathbb{N}$ .

### 2.3 The semantics

We introduce a set theoretic semantics to define the meanings of syntactic structures and knowledge. An *interpretation* is a pair  $\langle \Delta, .^I \rangle$ , where  $\Delta$  is a domain of elements that admits naive set theory and  $.^I$  is a mapping function that maps individuals into domain elements in  $\Delta$ , concepts into sets in  $\Delta$  and operators into functions in  $\Delta$ . The mapping functions  $.^I$  can be generalized into mapping from terms to elements.

Let  $I$  be an interpretation and  $a = b$  an assertion. We say that  $I$  is a *model* of  $a = b$ , denoted by  $I \models a = b$  iff  $.^I(a) = .^I(b)$ , also written  $a^I = b^I$ . Let  $KB$  be a knowledge base. We say that  $I$  is a model of  $KB$ , denoted by  $I \models KB$ , iff  $I$  is a model of every assertion in  $KB$ . We say that an assertion  $A$  is a *property* of  $KB$ , denoted by  $KB \models A$ , iff for all interpretations  $I$  such that  $I \models KB$ , we have  $I \models A$ . In particular, we say that an assertion  $A$  is a *tautology* iff it is modeled by all interpretations.

Since we assume naive set theory, we directly borrow some set theoretic constructs on individuals, concepts and operators. For instance, we can use  $\cup(C_1, C_2)$  (also written as  $C_1 \cup C_2$ ) to denote a new concept that unions two concepts  $C_1$  and  $C_2$ . Applying this to assertions, we can see that assertions of the form (1) can indeed represent many important features in knowledge representation. For instance, the *membership assertion*, stating that an individual  $a$  is an instance of a concept  $C$  is the following assertion  $\in(a, C) = \top$  (also written as  $a \in C$ ). The *containment assertion*, stating that a concept  $C_1$  is contained by another concept  $C_2$ , is the following assertion  $\subseteq(C_1, C_2) = \top$  (also written as  $C_1 \subseteq C_2$ ). The *range declaration*, stating that the range of an operator  $O$  operating on some concept  $C_1$  equals to another concept  $C_2$  is the following assertion  $O(C_1) = C_2$ .

## 3 Definitions for Extensibility

The primitive form is a foundation that uses minimal assumed knowledge and primitive constructs. Nevertheless, sometimes it is not convenient to use it for formalizing an application domain, e.g., to represent logic expressions. Hence, we extend it with more

building blocks. As discussed in the introduction section, extensibility is a critical issue for KR approaches.

For this purpose, we introduce *definitions* in our approach. Definitions use (schema) assertions to define new syntactic objects (individuals, concepts and operators) based on existing ones. Note that definitions are nothing extra but special kinds of knowledge.

### 3.1 Defining individuals, operators and concepts

We start with defining new individuals. An individual definition is a special kind of assertion of the form

$$a = t, \quad (2)$$

where  $a$  is an atomic individual and  $t$  is a term. Here,  $a$  is the individual to be defined. This assertion claims that the left side  $a$  is defined as the right side  $t$ . For instance,  $0 = \emptyset$  means that the individual 0 is defined as the empty set.

Defining new operators is similar to defining new individuals except that we use schema assertions for this purpose. Let  $O$  be an operator to be defined and  $(C_1, \dots, C_n)$  its domain. An operator definition is a schema assertion of the form

$$O(C_1, \dots, C_n) = T, \quad (3)$$

where  $T$  is a schema term that mentions concepts only from  $C_1, \dots, C_n$ . It could be the case that  $T$  only mentions some of  $C_1, \dots, C_n$ . Note that if  $C_1, \dots, C_n$  refer to the same concept, we need to use different copies respectively.

Since a schema assertion represents a set of assertions, essentially, an operator definition of the form (3) defines the operator  $O$  by defining the value of  $O(a_1, \dots, a_n)$  one-by-one, where  $a_i \in C_i, 1 \leq i \leq n$ . Sometimes we also define operators in this way. For instance, for defining the successor operator  $Succ$ , we can use the schema assertion  $Succ(\mathbb{N}) = \{\mathbb{N}, \{\mathbb{N}\}\}$ . This is equivalent to an alternative definition stating that, for every natural number  $n$ , the successor of  $n$ , is defined as  $\{n, \{n\}\}$ , i.e.,  $Succ(n) = \{n, \{n\}\}$ . For instance,  $Succ(0)$  is defined as  $\{\emptyset, \{\emptyset\}\}$ .

Defining new concepts is different. As concepts are essentially sets, they are defined through set theoretic constructions. We directly borrow set theory notations to define concepts as follows:

**Enumeration** Let  $a_1, \dots, a_n$  be  $n$  individuals. Then, the collection  $\{a_1, \dots, a_n\}$  is a concept, written as

$$C = \{a_1, \dots, a_n\}. \quad (4)$$

For instance, we can define the concept *Digits* by  $Digits = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ .

**Operation** Let  $C_1$  and  $C_2$  be two concepts. Then,  $C_1 \cup C_2$  (the union of  $C_1$  and  $C_2$ ),  $C_1 \cap C_2$  (the intersection of  $C_1$  and  $C_2$ ),  $C_1 \setminus C_2$  (the difference of  $C_1$  and  $C_2$ ),  $C_1 \times C_2$  (the Cartesian product of  $C_1$  and  $C_2$ ),  $2^{C_1}$  (the power set of  $C_1$ ) are concepts. Operation can be written by assertions as well. For instance, the following assertion

$$C = C_1 \cup C_2 \quad (5)$$

states that the concept  $C$  is defined as the union of  $C_1$  and  $C_2$ . As an example, one can define the concept *Man* by  $Man = Human \cap Male$ .

**Comprehension** Let  $C$  be a concept and  $A(C)$  a schema assertion that only mentions concept  $C$ . Then, individuals in  $C$  satisfying  $A$ , denoted by  $\{x \in C \mid A(x)\}$  (or simply  $C \mid A(C)$ ), form a concept, written as

$$C' = C \mid A(C). \quad (6)$$

For instance, we can define the concept *Male* by  $Male = \{Animal \mid Sex(Animal) = male\}$ , meaning that *Male* consists of all animals whose sex are male.

**Replacement** Let  $O$  be an operator and  $C$  a concept on which  $O$  is well defined. Then, the individuals mapped from  $C$  by  $O$ , denoted by  $\{O(x) \mid x \in C\}$  (or simply  $O(C)$ ), form a concept, written as

$$C' = O(C). \quad (7)$$

For instance, we can define the concept *Parents* by  $Parents = ParentOf(Human)$ , meaning that it consists of all individuals who is a *ParentOf* some human.

Definitions can be incremental. We may define some syntactic objects first. Once defined, they can be used to define more. One can always continue with this incremental process to extend the current system. For instance, in arithmetic, we define the successor operator first. Once defined, it can be used to define the add operator, which is further served as a basis to define more and more useful syntactic objects.

For clarity, we use the symbol “ $::=$ ” to replace “ $=$ ” for definitions. We force uniqueness of definitions. That is, each syntactic object can only be defined at most once.

Another critical issue is about recursiveness. Clearly, a definition such as  $a ::= a + 1$  is invalid and meaningless. Hence, we need to restrict our definitions. However, sometimes we do use recursion to define concepts. For instance, in arithmetic, natural numbers are define as  $\mathbb{N} ::= \{0\} \cup Succ(\mathbb{N})$ , meaning that if  $n$  is a natural number, then the successor of  $n$ , i.e.  $Succ(n)$ , is also a natural number.

We require that recursion can only be used in replacement definition of concepts. When a recursive replacement is used, we interpret it as an infinite process. At the beginning, all concepts contain and only contain those individuals defined by non-replacement definitions. Then, we apply the replacement definitions to obtain new versions of concepts. This finishes the first step. We continue with the process. At each step, we first use those non-replacement definitions to expand the concepts. Then, again, we apply the replacement definitions to obtain new versions of concepts. This could be an infinite process. For instance, consider the definition of natural numbers. Initially, we have  $\{0\}$ . Then, applying the replacement definition, we expand it to  $\{0, Succ(0)\}$ . We continue with this process to obtain the infinite set of natural numbers  $\{0, Succ(0), Succ(Succ(0)), \dots\}$ .

We require that all other definitions are non-recursive. Formally, the *definition dependency graph* over a set of definitions (without replacements) is a directed graph  $\langle V, E \rangle$ , where  $V$  consists of all syntactic objects appeared in these definitions and  $E$  is the set of all pairs  $(a, b)$  such that there exists a definition whose left side is  $a$  and whose right side mentions  $b$ .<sup>2</sup> A set of definitions is said to be *non-recursive* if its corresponding definition dependency graph is acyclic.

<sup>2</sup>For operator definitions, we ignore the concepts that are arguments in the operator and in the schema term since they are essentially grounded into individuals.

In fact, one can observe that the Backus-Naur form (BNF), widely used in computer science to define syntax, can be considered as a special case of our concept definitions over strings. More precisely, BNF only uses three features, enumeration (of a single element), union operation and recursive replacement by using the pre-assumed concatenation operator. Comprehension and other set operations are not used.

### 3.2 Multi-assertions

As a case study of extending the primitive form by definitions, we extend assertions of the form (1) into multi-assertions.

Given a number  $n$ , we define a new operator  $M_n$  for multi-assertions with arity  $n$  by the following schema assertion:

$$M_n(C_1 = D_1, \dots, C_n = D_n) ::= (C_1, \dots, C_n) = (D_1, \dots, D_n),$$

where  $C_i, D_i, 1 \leq i \leq n$ , are concepts of terms. This assertion states that for  $n$  assertions,  $Assertion_i, 1 \leq i \leq n$ , of the form (1), namely  $a_i = b_i$ ,  $M_n(a_1 = b_1, \dots, a_n = b_n)$  is  $(a_1, \dots, a_n) = (b_1, \dots, b_n)$ . Hence,  $M_n(a_1 = b_1, \dots, a_n = b_n)$  holds if and only if for all  $i, 1 \leq i \leq n$ ,  $a_i = b_i$ , that is,  $Assertion_i$  holds. In this sense, this single assertion can be used to represent  $n$  assertions  $Assertion_i, 1 \leq i \leq n$ .

Then, we define the concept of multi-assertions as follows:

$$Multi - Assertion ::= \bigcup_{1 \leq i \leq \infty} M_i(\mathcal{A}^1, \dots, \mathcal{A}^i),$$

where  $\mathcal{A}^1, \dots, \mathcal{A}^i$  are  $i$  copies of standard assertions. For convenience, we use  $Assertion_1, \dots, Assertion_n$  to denote a  $n$ -ary multi-assertion.

Note that multiple assertions are just syntactic sugar of the primitive form as they can be defined in the primitive form by using ordered pairs and Cartesian products. In this sense, they do not increase the expressive power of the primitive form. Nevertheless, using them can make the representation task more convenient in some cases. Multi-assertions are not only of interests themselves. Once defined, they can be used to define more syntactic building blocks. Note that finite knowledge bases are finite sets of assertions, i.e., multi-assertions, which can essentially be viewed as single assertions.

### 3.3 Nested terms and assertions

We continue with our extensions for the primitive form by introducing nested terms and nested assertions. Note that terms defined in Section 2 cannot be nested in the sense that individuals used inside an operator must be atomic. This can be generalized to nested terms, where operators can use compound individuals inside.

Nested terms are defined by the following definition:

$$\begin{aligned} Nested - Term & ::= Term \cup N - Term \\ N - Term & ::= Op(Nested - Term), \end{aligned}$$

where  $Term$  is the concept of standard term defined in Section 2,  $Op$  is an arbitrary operator and  $Op(Nest - Term)$  is a replacement definition such that individuals in  $Nest - Term$  are in the domain of  $Op$ .



The above definition is a recursive definition. In fact, it can be simplified as

$$Nested - Term ::= Term \cup Op(Nested - Term).$$

However, this definition itself uses nested terms as well since  $Op(Nested - Term)$  is not atomic. Hence, before formally defining the meaning of nested terms, we use the former.

Nevertheless, from this example, we can see how to interpret nested terms. That is, whenever a nested term is used, we introduce a new atomic individual to replace it, and claim that this atomic individual defines the nested term. To formalize this idea, we also need nested assertions, in which terms used on both sides of the assertion can be nested.

$$Nested - Assertion ::= Nested - Term = Nested - Term.$$

As mentioned, nested assertions can be represented by non-nested multi-assertions by introducing new individuals. Whenever a result of nested term is used, we introduce a new individual to replace it and claim that this new individual is defined as the nested term. That is, for every nested term  $Op(a_1, \dots, Op'(b_1, \dots, b_m), \dots, a_m)$  occurred in a nested assertion, we introduce a new atomic individual  $a'$ ; replace the above term with  $Op(a_1, \dots, a', \dots, a_m)$  and add a new assertion  $a' = Op'(b_1, \dots, b_m)$ . For instance, the nested assertion  $Op(a, Op(b, Op'(c))) = Op'(d)$  is defined as  $Op(a, x) = Op'(d), x = Op(b, y), y = Op'(c)$ , where  $x$  and  $y$  are new individuals. In this sense, nested assertion is essentially a multi-assertion, which can be represented as a single assertion. Therefore, nested assertion is a syntactic sugar of the primitive form as well.

Using nested assertions can simplify the representation task. However, one cannot overuse nested assertions since, essentially, every use of a nested term introduces a new individual. For instance, one can easily get lost with a nested assertion like  $Op(a, Op(b, Op'(c))) = Op'(d)$ .

## 4 Logic Operators over Assertions

In this section, we continue to extend the primitive form with logic operators over assertions. Interestingly, we can define not only propositional connectives but also quantifiers based on naive set theory. This, on one side, provides another case study how to extend the primitive form, and on the other side, shows that, assuming naive set theory, the primitive form is expressive enough to capture logic.

### 4.1 Propositional operators over assertions

We start with the propositional case. Let  $\mathcal{A}$  be the concept of nested assertions. We introduce a number of operators over  $\mathcal{A}$ , including  $\neg(\mathcal{A})$  (for *negation*),  $\wedge(\mathcal{A}^1, \mathcal{A}^2)$  (for *conjunction*),  $\vee(\mathcal{A}^1, \mathcal{A}^2)$  (for *disjunction*),  $\rightarrow(\mathcal{A}^1, \mathcal{A}^2)$  (for *implication*) and  $\equiv(\mathcal{A}^1, \mathcal{A}^2)$  (for *equivalence*).

There could be different ways to define those operators, depending on which operators are defined directly and which are defined based on the previous ones. Here, we



directly define negation, conjunction, disjunction and implication and indirectly define equivalence.

Let  $a = a'$  and  $b = b'$  be two (nested) assertions. The propositional connectives are defined as follows:

$$\begin{aligned} \neg(a = a') &::= & \{a\} \cap \{a'\} &= \emptyset \\ \wedge(a = a', b = b') &::= & (\{a\} \cap \{a'\}) \cup (\{b\} \cap \{b'\}) &= \{a, a', b, b'\} \\ \vee(a = a', b = b') &::= & (\{a\} \cap \{a'\}) \cup (\{b\} \cap \{b'\}) &\neq \emptyset \\ \rightarrow(a = a', b = b') &::= & (\{a, a'\} \setminus \{a\} \cap \{a'\}) \cup (\{b\} \cap \{b'\}) &\neq \emptyset \\ \equiv(a = a', b = b') &::= & \wedge(\rightarrow(a = a', b = b'), \rightarrow(b = b', a = a')). \end{aligned}$$

We also use  $a \neq a'$  to denote  $\neg(a = a')$ . One can observe that the ranges of all logic operators are nested assertions. Hence, similar to multi- and nested assertions, propositional logic operators are syntactic sugar as well.

Now we consider some properties. For instance, according to the definitions, De-Morgan's laws are tautologies.

**Theorem 1.** *Let  $A_1$  and  $A_2$  be two (nested) assertions. Then, for all interpretations  $I$ ,*

$$I \models \neg(A_1 \vee A_2) \equiv \neg(A_1) \wedge \neg(A_2).$$

$$I \models \neg(A_1 \wedge A_2) \equiv \neg(A_1) \vee \neg(A_2).$$

Also, the relationship between implication and disjunction, i.e.,  $A_1 \rightarrow A_2 \equiv \neg A_1 \vee A_2$ , is a tautology as well.

In fact, all tautologies in propositional logic are tautologies under our context, i.e., modeled by all interpretations, and vice versa. This, actually follows from the following theorem, stating that the syntactic definitions above defines the semantics of logic operators.

**Theorem 2.** *Let  $A_1$  and  $A_2$  be two nested assertions. Then, for all interpretations  $I$ ,*

- $I \models \neg(A_1)$  iff  $I$  is not a model of  $A_1$ .
- $I \models \wedge(A_1, A_2)$  iff  $I$  is a model of both  $A_1$  and  $A_2$ .
- $I \models \vee(A_1, A_2)$  iff  $I$  is a model of either  $A_1$  or  $A_2$ .
- $I \models \rightarrow(A_1, A_2)$  iff  $I$  is a model of  $A_1$  implies that  $I$  is a model of  $A_2$ .
- $I \models \equiv(A_1, A_2)$  iff  $I$  is a model of  $A_1$  if and only if  $I$  is a model of  $A_2$ .

## 4.2 Quantifiers over assertions

Now we consider quantifiers, including  $\forall$  (for the *universal* quantifier) and  $\exists$  (for the *existential* quantifier). The domain of quantifiers is a pair  $(C, A(C))$ , where  $C$  is a concept and  $A(C)$  is a schema assertion that only mentions  $C$ .

The quantifiers are defined as follows:

$$\begin{aligned} \forall(C, A(C)) &::= & C|A(C) &= C \\ \exists(C, A(C)) &::= & C|A(C) &\neq \emptyset \end{aligned}$$

Intuitively,  $\forall(C, A(C))$  is true iff those individuals  $x$  in  $C$  such that  $A(x)$  holds equals to the concept  $C$  itself, that is, for all individuals  $x$  in  $C$ ,  $A(x)$  holds;  $\exists(C, A(C))$  is true iff those individuals  $x$  in  $C$  such that  $A(x)$  holds does not equal to the empty set, that is, there exists at least one individual  $x$  in  $C$  such that  $A(x)$  holds. We can see that the ranges of quantifiers are nested assertions as well. Thus, quantifiers are also syntactic sugar of the primitive form.

Similarly, the syntactic definitions of quantifiers based on naive set theory capture their semantics.

**Theorem 3.** *Let  $C$  be a concept and  $A(C)$  a schema assertion that only mentions  $C$ . For all interpretations  $I$ ,*

- $I \models \forall(C, A(C))$  iff for all individuals  $a$  in  $C$ ,  $I \models A(a)$ .
- $I \models \exists(C, A(C))$  iff there exists at least one individual  $a$  in  $C$  such that  $I \models A(a)$ .

As a consequence, one can prove some properties about quantifiers. For instance, the universal quantifiers and the existential quantifiers are dual under negation.

Note that quantifiers defined here are ranging from an arbitrary concept  $C$ . If  $C$  is a concept of all atomic individuals and all quantifiers range from the same concept  $C$ , then these quantifiers are first-order. Nevertheless, the concepts could be different. In this case, we have many-sorted first-order logic. Moreover,  $C$  could be complex concepts, e.g., a concept of all possible concepts. In this case, we have monadic second-order logic. Yet  $C$  could be many more, e.g., a concept of assertions, a concept of concepts of terms etc. In this sense, the quantifiers become high-order.

## 5 Conclusions, Discussions and Related Work

In this paper, we have proposed a set theoretic approach to syntactically represent knowledge in application domains. The syntax of a domain is captured by individuals (i.e., objects in the domain), concepts (i.e., groups of objects sharing something in common) and operators (i.e., connections and relationships among objects). From a set theory point of view, individuals, concepts and operators are interpreted as elements, sets and functions respectively. In the primitive form, knowledge in the domain is simply captured by equality assertions of the form  $a = b$ , where  $a$  and  $b$  are terms.

We have shown how to extend a system by definitions, which are special kinds of knowledge used to define new individuals, concepts and operators. For instance, we have extended the primitive form with multi-assertions and nested assertions, which are just syntactic sugar of the primitive form as they can be expressed in it. Extensibility is a critical issue for KR. A KR approach should be able to define new syntactic objects based on exiting ones. Once defined, these objects can be further used to define more.

Interestingly, we have shown that logic operators, not only propositional connectives but also quantifiers, can be defined in the primitive form based on naive set theory. This, on one side, shows that we can define the semantics of logic operators syntactically, and on the other side, shows the expressiveness of our approach.

As discussed in the introduction section, our motivation is to propose a simple, extensible, expressive and efficient KR approach. While extensibility and expressiveness are discussed in the above two paragraphs, simplicity is difficult to justify. We argue that our approach indeed satisfy the three aspects of simplicity. For primitiveness, our approach only uses naive set theory, syntax including individuals, concepts and operators and knowledge of the form  $a = b$ . For succinctness, the primitive form only needs at most double length to simulate logic, which is an arguably succinct KR formalism. For user-friendliness, we believe that knowledge of the form  $a = b$ , similar to the assignment statement in programming, can be easily understood and used by knowledge engineers.

Certainly, one can define multi-assertion, nested assertion and logic operators directly. Nevertheless, our motivation is to provide a simple foundation for knowledge representation so that all other features and building blocks in KR can be defined as extensions of it. We believe that our primitive form is such a candidate, evident from the fact that it is able to capture high-order logic expressions.

This work has two philosophical implications. First, for answering the question “what is knowledge”, our approach defines it as equality assertions between two terms. Again, evident from the fact that single equality assertions can capture high-order logic expressions based on naive set theory, we believe that it provides a uniformed view of what knowledge is. Such a uniformed view is critical for not only understanding and representing knowledge but also utilizing and reasoning about knowledge. Second, we have shown that we can define the semantics of logic syntactically based on naive set theory. We believe that the same thing can be done for the semantics of other features in KR, e.g., nonmonotonic reasoning. This is useful as most operations done by machines are syntax based.

This paper is mainly focused on the representation part. We leave the reasoning part and the efficiency discussions to another paper. Nevertheless, it is worth mentioning a little here. Roughly speaking, reasoning is about how to derive properties from a knowledge base. We distinguish between querying and reasoning. The former is to check whether an assertion is a property of a knowledge base, while the latter is to find some properties of a given knowledge base. Clearly, reasoning can serve as a means for querying, but they are not the same. Querying is generally difficult of our approach as it can express logic. Nevertheless, reasoning could be efficient, and that is exactly the focus of our reasoning paper.

Although querying the full language is generally undecidable, there could be some meaningful and useful tractable subclasses. An important case is database. Note again that, in the primitive form, knowledge is simply formalized by equality assertions of the form  $a = b$ . Nevertheless, this can be indeed expressive as  $a$  and  $b$  could be complex nested terms. A database under our context only contains two kinds of equality assertions, i.e., data of the form  $Op(a_1, \dots, a_n) = b$ , where  $a_i, 1 \leq i \leq n$ , and  $b$  are atomic individuals and membership statements  $a \in C$ , where  $a$  is an atomic individual and  $C$  is an atomic concept. In this sense, data is a special kind of knowledge. Querying on such a database is tractable. We leave the detailed discussions to another work.

Our set theoretic KR approach is deeply inspired by and rooted in many other KR formalisms, including propositional and first-order logic, semantic network and description logic. The dynamic version of this approach (will be presented in another

paper) is deeply related to rule based formalisms including Hoare logic and answer set programming. Interestingly, although originated from a different motivation, our approach shares many basic ideas and borrows many notations from description logic [Baader *et al.*, 2003]. In fact, we can rewrite all building blocks in description logic to our approach since the primitive form can capture first-order logic. Table 1 depicts some of them. Here,  $\hat{R}$  is defined to transform a binary Boolean relationship  $R$  to a

Table 1: Rewriting description logic into our approach

Constructs	Description logic	Our approach
individual	individual	individual
concept	concept	concept
role	Role	binary Boolean operator
intersection	$C \sqcap D$	$C \cap D$
union	$C \sqcup D$	$C \cup D$
complement	$\neg C$	$\mathcal{I} \setminus C$
reverse role	$R^-$	$R^-(C, D) ::= R(D, C)$
existential restriction	$\exists R.C$	$\mathcal{I} \hat{R}^-(\mathcal{I}) \cap C \neq \emptyset$
universal restriction	$\forall R.C$	$\mathcal{I} \hat{R}^-(\mathcal{I}) \subseteq C$
at least restriction	$\geq nR.C$	$\mathcal{I} (R^-(\mathcal{I}) \cap C)^C \geq n$
nominal	$\{a\}$	$\{a\}$
concept assertion	$C(a)$	$a \in C$
role assertion	$R(a, b)$	$R(a, b)$
individual equality	$a \approx b$	$a = b$
concept inclusion	$C \sqsubseteq D$	$C \subseteq D$

unary operator, i.e.,  $\hat{R}(C) ::= D|R(C, D)$ , and  $(D)^C$  denotes the cardinality of  $D$ .

Nevertheless, our approach differs from description logic in several essential ways. The most important difference is that, for the purpose of forming new concepts by operators/roles, our approach directly uses set theoretic constructs including comprehension and replacement, while description logics use role restrictions. As an example, suppose that we want to specify a concept including all human having female children. In our approach, this is formalized by  $Human \mid Children(Human) \cap Female \neq \emptyset$ , while in description logic, it is formalized by  $\exists Parentof.(Female)$ . Secondly, we use multi-ary operators, e.g., the *Add* operator, instead of binary Boolean relationships to connect individuals/concepts. Thirdly, all knowledge in our approach are essentially formalized in the same form, i.e., equality assertions. Fourthly, we allow complex assertions including high-order constructs. Last but not least, we particularly highlight the importance of extensibility in our approach.

We shall present a series of papers to propose the set theoretic knowledge representation approach. This paper is a foundation that is mainly concerned with the basic ideas and the representation part. As mentioned, there are a number of things to present in the future. One critical task is to present the reasoning part. Another one is to formalize dynamics, including how to represent basic and compound actions, how to describe the effects of actions and the interactions among knowledge and actions.

## References

- [Baader *et al.*, 2003] Franz Baader, Diego Calvanese, Deborah L. McGuinness, Daniele Nardi, and Peter F. Patel-Schneider, editors. *The Description Logic Handbook: Theory, Implementation, and Applications*. Cambridge University Press, New York, NY, USA, 2003.
- [Baral, 2003] Chitta Baral. *Knowledge Representation, Reasoning, and Declarative Problem Solving*. Cambridge University Press, New York, NY, USA, 2003.
- [Brachman and Levesque, 2004] Ronald J. Brachman and Hector J. Levesque. *Knowledge Representation and Reasoning*. Elsevier, 2004.
- [Halmos, 1960] Paul Halmos. *Naive Set Theory*. Van Nostrand, 1960. Reprinted by Springer-Verlag, Undergraduate Texts in Mathematics, 1974.
- [Levesque and Brachman, 1987] Hector J. Levesque and Ronald J. Brachman. Expressiveness and tractability in knowledge representation and reasoning. *Computational Intelligence*, 3:78–93, 1987.
- [van Harmelen *et al.*, 2008] Frank van Harmelen, Vladimir Lifschitz, and Bruce W. Porter, editors. *Handbook of Knowledge Representation*, volume 3 of *Foundations of Artificial Intelligence*. Elsevier, 2008.